Fermion Pair Production in Planar Coulomb and Aharonov-Bohm Potentials

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Exact analytic solutions are found for the Dirac equation in 2+1 dimensions for a spin-one-half particle in a combination of the Lorentz 3-vector and scalar Coulomb as well as Aharonov-Bohm potentials. We employ the two-component Dirac equation which contains a new parameter introduced by Hagen to describe the spin of the spin-1/2 particle. We derive transcendental equations that implicitly determine the energy spectrum of an electron near the negative-energy continuum boundary and the critical charges for some electron states. Fermion pair production from a vacuum by a strong Coulomb field in the presence of the magnetic flux tube of zero radius is considered. It is found that the presence of the Ahanorov-Bohm flux tends to stabilize the system.

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I. INTRODUCTION

The Aharonov-Bohm (AB) effect is one of the most intriguing effects of a truly quantal nature [1]. Ever since its discovery, the AB effect has been analyzed in various physical situations in numerous works [2]. In recent years there has been considerable interest in the problem of the scattering of a spin-1/2 particle off an AB potential in 2+1 dimensions. The results of [1] for the nonrelativistic case modified by using the Dirac equation in 2+1 dimensions were applied to many problems. For instance, solutions to the two-component Dirac equation in the AB potential were first discussed by Alford and Wilczek in [3] in a study of the interaction of cosmic strings with matter. In particular, a mechanism of particle production due to the nonstatic AB potential of a moving cosmic string was discussed in [3]. The relativistic quantum AB effect was studied in Ref. [4] for the free and bound fermion states by means of exact analytic solutions of the Dirac equation in 2+1 dimensions for a combination of AB, Lorentz three-vector, and scalar Coulomb potentials.

In [5] the scattering of spin-polarized fermions in an AB potential was considered in 2+1 dimensions. There the particle spin was introduced into the two-component Dirac equation as a new parameter. The term including this new parameter appears in the form of an additional delta-function interaction of spin with the magnetic field in the Dirac equation. Solutions of the Dirac equation were then interpreted for the case of 3+1 dimensions. Similar problems were also discussed in [6] by taking the AB flux tube with a small but finite radius in the (3+1)-dimensional Pauli equation and in the (2+1)-dimensional Dirac equation.

In this paper we study how various physical fields affect the properties of a bound Dirac fermion in 2+1 dimensions. Specifically, we study the energy spectrum of the fermion in a combination of the Lorentz 3-vector and scalar Coulomb as well as AB potentials. We also consider the influence of the magnetic flux tube of zero radius on the so-called critical charge which determines the onset of instability of the system, and on the probability of production of an electron-positron pair from a vacuum by a strong vector Coulomb field. We note here that in 3+1 dimensions analytic solutions of such problems, even for the Schrödinger equation in the Coulomb and AB potentials, have not yet been found.

This paper is organized as follows. In Section II we find the exact bound states solutions of the Dirac equation in 2+1 dimensions for a combination of the Lorentz 3-vector, scalar Coulomb, and AB potentials for the spin-1/2 particle. The formulas for the eigen-energies of the relativistic fermion are obtained and discussed. In Section III the critical charge and the probability of production of an electron-positron pair from a vacuum by a strong vector Coulomb field are calculated. Using a simplified model with a truncated Coulomb potential, we show that the critical charge and the production probability of a pair are influenced by the magnetic flux and the spin of the particle.

II. ENERGY SPECTRUM OF A DIRAC FERMION IN PLANAR COULOMB AND AHARONOV-BOHM POTENTIALS

The (2+1)-dimensional Dirac equation of a fermion of mass m and charge $e = -e_0 < 0$ in a vector potential A_μ and a Lorentz scalar potential U is ($c = \hbar = 1$)

$$(\gamma^\mu P_\mu - m - U)\Psi = 0, \tag{1}$$

where $P_\mu = -i\partial_\mu - eA_\mu$ is the generalized fermion momentum operator. The Dirac γ^μ matrices are conveniently defined in terms of the Pauli spin matrices as

$$\gamma^0 = \sigma_3, \quad \gamma^1 = is\sigma_1, \quad \gamma^2 = i\sigma_2, \tag{2}$$

Following [5], here s is a new parameter characterizing twice the spin value $s = \pm 1$ for spin "up" and "down", respectively.

We are interested in finding exact analytic solutions of the Dirac equation for both signs of s in an AB potential, which is specified in Cartesian or cylindrical coordinates as

$$A^0 = 0, \quad A_x = -\frac{By}{r^2}, \quad A_y = \frac{Bx}{r^2}; \quad A^0 = 0, \quad A_r = 0, \quad A_\varphi = \frac{B}{r},$$
$$r = \sqrt{x^2 + y^2}, \quad \varphi = \tan^{-1}(y/x), \tag{3}$$

and a Lorentz 3-vector potential $(A^\mu(r))$ and a scalar potential $(U(r))$ potential defined by

$$A^0(r) = \frac{a}{|e|r}, \quad A_r = 0, \quad A_\varphi = 0; \quad U(r) = -\frac{b}{r} \quad (a, b > 0). \tag{4}$$

In [4] only the case for $s = 1$ was considered. Note that if $e_0 B = N$, where N is an integer, then the magnetic field flux is quantized as $\Phi = \Phi_0 N$, where $\Phi_0 \equiv 2\pi/e_0$ is the elementary magnetic flux called the "fluxon".

The Dirac Hamiltonian for this system is

$$H_D = \sigma_1 P_2 - s\sigma_2 P_1 + \sigma_3(m + U(r)) - e_0 A_0(r). \tag{5}$$

The vector potential A_μ in the generalized momentum P_μ is the sum of the AB potential (3) and the Lorentz 3-vector potential (4). The total angular momentum is $J_z \equiv L_z + s\sigma_3/2$, where $L_z \equiv -i\partial/\partial\varphi$ is a conserved quantity.

We seek positive energy solutions of Eq. (1) in the form (see also Ref. [7–9])

$$\Psi(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \exp(-iEt + il\varphi)\psi(r, \varphi) \ , \tag{6}$$

where $E \geq 0$ is the fermion energy, l is an integer, and $\psi(r, \varphi)$ is a two-component function (i.e., a 2-spinor)

$$\psi(r, \varphi) = \begin{pmatrix} f(r) \\ g(r)e^{is\varphi} \end{pmatrix} . \tag{7}$$

The wave function Ψ is an eigenfunction of the conserved total angular momentum J_z with eigenvalue $j = l + s/2$.

From the Dirac equation one finds that $f(r)$ and $g(r)$ satisfy the following equations:

$$\begin{aligned} s\frac{df}{dr} - \frac{l + e_0 B}{r} f + \left(E + m + \frac{a-b}{r}\right) g = 0, \\ s\frac{dg}{dr} + \frac{l + s + e_0 B}{r} g - \left(E - m + \frac{a+b}{r}\right) f = 0. \end{aligned} \tag{8}$$

Compared with the situation studied in [8] in which only the vector Coulomb potential (a) is present, one notes that the effect of the AB potential (B) appears only in modifying the angular momentum l. As in [8], we now assume $f(r)$ and $g(r)$ to have the form (see, for example, [10])

$$\begin{aligned} f(r) &= \sqrt{m+E}\, e^{-x/2} x^{\gamma_s} (Q_1^s + Q_2^s) \ , \\ g(r) &= \sqrt{m-E}\, e^{-x/2} x^{\gamma_s} (Q_1^s - Q_2^s) \ , \end{aligned} \tag{9}$$

where

$$x = 2\lambda r, \quad \lambda = \sqrt{m^2 - E^2}, \tag{10}$$

and

$$\gamma_s = -\frac{1}{2} \pm \sqrt{\left(l + e_0 B + \frac{s}{2}\right)^2 - a^2 + b^2} \tag{11}$$

determines the asymptotic behavior of the wave function for small r.

From Eq. (11) one sees that when $a^2 < (l + e_0 B + s/2)^2 + b^2$ the quantity γ_s is real and must be chosen positive to ensure normalizability of the wave function. If $a^2 > (l + e_0 B + s/2)^2 + b^2$ then the two roots of γ_s are imaginary, and the corresponding wave functions oscillate as $r \to 0$, which indicates the occurrence of Klein's paradox [10, 11]. We shall consider this situation in the next section. In what follows we shall take

$$\gamma_s = -\frac{1}{2} + \sqrt{\left(l + e_0 B + \frac{s}{2}\right)^2 - a^2 + b^2}. \tag{12}$$

For wave functions that are finite at $x = 0$, the functions $Q^s_{1,2}$ for $s = \pm 1$ are given by the confluent hypergeometric function $F(a_1, c_1; x)$:

$$\begin{aligned} Q^s_1 &= AF\left(\gamma_s + 1 - \frac{s}{2} - \frac{aE + mb}{\lambda}, 2\gamma_s + 2; x\right), \\ Q^s_2 &= CF\left(\gamma_s + 1 + \frac{s}{2} - \frac{aE + mb}{\lambda}, 2\gamma_s + 2; x\right). \end{aligned} \tag{13}$$

The constants A and C are related by

$$C = \frac{(s\gamma_s + s/2) - (Ea + mb)/\lambda}{l + e_0 B + s/2 + (ma + bE)/\lambda} A. \tag{14}$$

The wave function is normalizable if both the hypergeometric functions Q^s_1 and Q^s_2 are reduced to polynomials. For this the parameter a of $F(a, c; x)$ must be a negative integer or zero. Denoting

$$\gamma_s + 1 - \frac{s}{2} - \frac{Ea + mb}{\lambda} = -n_r, \tag{15}$$

one can see that if $n_r = 1, 2, 3, \ldots$, then Q^1_1 and Q^1_2 are reduced to polynomials for $s = 1$. If $n_r = 0$, then only Q^1_1 is reduced to a polynomial. But the relation $n_r = 0$ implies that

$$\gamma_1 + \frac{1}{2} = \frac{Ea + mb}{\lambda} > 0. \tag{16}$$

Then it follows from (16) and the relation

$$\left(\gamma_s + \frac{1}{2}\right)^2 - \frac{a^2 - b^2}{\lambda^2} E^2 = \left(l + e_0 B + \frac{s}{2}\right)^2 - \frac{a^2 - b^2}{\lambda^2} m^2 \tag{17}$$

that

$$\frac{Eb + ma}{\lambda} = \left| l + e_0 B + \frac{1}{2} \right|. \tag{18}$$

If $l + e_0 B + 1/2 > 0$ then $C = 0$, hence $Q^1_2 = 0$ and the required condition is not violated (Q^1_1 is a polynomial). If $l + e_0 B + 1/2 < 0$ then $C = A$, and Q^1_2 is still a divergent function. The following values of n_r are hence admissible: $0, 1, 2, \ldots$ for $l + e_0 B + 1/2 > 0$ and $1, 2, \ldots$ for $l + e_0 B + 1/2 < 0$.

For $s = -1$, Q^{-1}_1 and Q^{-1}_2 are reduced to polynomials if $n_r = 0, 1, 2, 3, \ldots$. But in this case n_r also can be equal to -1. If $n_r = -1$, then only Q^{-1}_2 is reduced to a polynomial. But for $n_r = -1$ from (15) one obtains

$$\gamma_{-1} + \frac{1}{2} = \frac{Ea + mb}{\lambda} > 0, \tag{19}$$

and Eq. (18) becomes

$$\frac{Eb+ma}{\lambda} = \left| l + e_0 B - \frac{1}{2} \right| . \tag{20}$$

It is convenient to rewrite the relation (14) for $s=-1$ as

$$A = -\frac{l + e_0 B - 1/2 + (ma+bE)/\lambda}{(\gamma_{-1}+1/2) + (Ea+mb)/\lambda} C. \tag{21}$$

So, if $l+e_0B-1/2<0$ then $A=0$, hence $Q_1^{-1}=0$ and the required condition is not violated (Q_2^{-1} is a polynomial). If $l+e_0B-1/2>0$ then $A=-C$, and Q_1^{-1} is divergent. Hence the following values of n_r are admissible for $s=-1$: $-1,0,1,2,\ldots$ for $l+e_0B-1/2<0$, and $0,1,2,\ldots$ for $l+e_0B-1/2>0$.

We can rewrite the equation (15) for the energy spectrum as

$$\frac{Ea+mb}{\lambda} = n_r + \gamma_s + 1 - \frac{s}{2} \equiv u, \tag{22}$$

from which we obtain finally the discrete fermion energy levels in the form

$$\frac{E_n}{m} = \sqrt{\left(\frac{ab}{u^2+a^2}\right)^2 + \frac{u^2-b^2}{u^2+a^2}} - \frac{ab}{u^2+a^2} . \tag{23}$$

One sees that the energy spectrum is influenced by the magnetic flux through γ_s given by Eq. (12). On the other hand, for the flux that is integer in the unit Φ_0, the energy spectrum is the same as in the absence of magnetic flux. The spectrum of the system changes only for flux that is not integer in the unit Φ_0. For such magnetic fluxes the energy spectrum is likely to be observed just as the AB effect. For flux that is not integer or half integer all the energy levels are doubly degenerate; the levels with l, n_r+1, $s=+1$ and $l+1$, n_r, $s=-1$ coincide. This reflects the fact that the fermion energy does not depend upon the spin in the field configuration considered.

If the scalar Coulomb potential is absent, then the energy spectrum is given by

$$E_{n,l} = m\left[1 + \frac{a^2}{(n_r + 1/2 - s/2 + \sqrt{(l+e_0B+s/2)^2 - a^2})^2}\right]^{-1/2} . \tag{24}$$

This expression makes sense only when $|l+e_0B+s/2|>a$, a condition that forbids the existence of the energy levels with $l+e_0B+s/2=0$.

In the nonrelativistic Schrödinger limit, the expression for the energy spectrum becomes

$$E_{non} = -\frac{a^2}{2(n_r+1/2-s/2+|l+e_0B+s/2|)^2} . \tag{25}$$

For flux e_0B that is not integer or half integer it has a full analogy with the Rydberg correction. Note that for flux that is half integer in the unit Φ_0 the energy spectrum

(25) depends only on the integer number. It can be observed using a spectroscope. It is interesting that for such magnetic fluxes the cross section in the AB scattering is maximal. It is seen that the eigen-energies of a fermion in these electromagnetic field combinations are periodic functions of the flux, like the case of the motion of a fermion in a closed ring in the absence of the two-dimensional Coulomb potential. The energy spectrum (25) repeats itself every time when the change in the flux e_0B is integral.

III. CRITICAL CHARGE AND PAIR PRODUCTION IN VECTOR COULOMB AND AHARONOV-BOHM POTENTIAL

We now consider the problem of stability of the system considered in the previous section when a becomes large. For simplicity we ignore the scalar Coulomb potential in this section, i.e., we set $b = 0$. Such consideration is relevant to the stability of the vacuum of quantum electrodynamics in a strong Coulomb field. In the absence also of the AB potential this problem in $3 + 1$ dimensions had been extensively studied in [12–18]. The corresponding system in $2 + 1$ dimensions was considered in [8, 9]. Now we would like to see how the presence of the AB potential may affect the stability of the system.

From Eq. (23) we see that the lowest electron states in the vector Coulomb and AB potentials are those with $n_r = l = 0$, $s = 1$, and $n_r = -1$, $l = 0$, $s = -1$. The electron energy in these states can be written as

$$E_s = m\sqrt{\frac{(e_0B + s/2)^2 - a^2}{(e_0B + s/2)^2}}\ . \tag{26}$$

For definiteness, in this section we shall consider positive flux ($e_0B > 0$). The case with negative flux can be discussed similarly with the signs of l and s flipped: it is just the mirror image of the case with positive flux with respect to the xy-plane. For magnetic flux which is positive and half integer in Φ_0, the energy of the state with $n_r = -1$, $l = 0$, $s = -1$ is divergent and imaginary for any value of a. So it is reliable to assume that the electron ground state in this case is the state with $n_r = l = 0$ and $s = 1$.

The energy $E_0(a)$ of this state as a function of a becomes zero at $a = a_0(B) \equiv e_0B + 1/2$ and purely imaginary when $a > a_0(B)$. This implies that $E_0(a)$ becomes meaningless at $a \geq a_0(B)$, and the wave function oscillates with infinite frequency at the origin as mentioned in Sect. II. It may appear that one cannot determine the spectrum beyond $a_0(B)$. However, one notes that in reality the source of the Coulomb field has finite extension, so that the potential remains finite at the origin. In this case the wave function is regular at the origin and the energy levels can be traced continuously beyond $a_0(B)$.

The problem of a (2+1)-dimensional Dirac particle in a strong field of a truncated (at small distances) Coulomb potential in the absence of an AB potential was considered in [8], and the expression for the electron energy spectrum was obtained in [9]. There it was shown that as the strength of the Coulomb source a increases, the lowest energy level (c.f., Eq. (26)) $E_{s=1}(B = 0) = m\sqrt{1 - 4a^2}$ was pulled towards the negative continuum. This energy becomes negative for $a^2 > 1/4$ (in the truncated potential) and may reach the negative-energy continuum boundary $-m$. When $E_{s=1}(B=0)$ dives into the negative continuum the vacuum of quantum electrodynamics becomes unstable and particle-antiparticle pairs are created spontaneously. The value $a = a_{cr}$ for which the lowest energy level coincides with $-m$ is called the critical charge for the ground state.

Let us consider the effect of the AB potential on the stability of the system qualitatively. From Eq. (8) we see that the effect of the AB potential appears only in increasing the angular momentum l. Classically it increases the centrifugal force on the particle. Thus it tends to counteract the tendency of the particle being pulled towards the negative continuum. Hence one expects that the AB potential (3) will stabilize the system against pair production.

Quantitatively, to determine the critical charge $a_{cr}(B)$ in the presence of the AB potential, it is sufficient to consider the range of electron energies near the negative-energy continuum limit $-m$. Introducing functions $F(r) = rf(r)$ and $G(r) = rg(r)$, and eliminating $G(r)$ from (8), we obtain the equation for the function $F(r)$ with $E \approx -m$ in the form

$$\frac{d^2F}{dr^2} + \left(E^2 - m^2 + \frac{2Ea}{r} + \frac{a^2 - (l+e_0B)(l+e_0B+1)}{r^2}\right) F = 0. \tag{27}$$

The solution for $G(r)$ near $E = -m$ can be found from the equation

$$G(r) = \frac{l+e_0B+1}{a} F(r) - s\frac{r}{a}\frac{dF}{dr}. \tag{28}$$

The solution of Eq. (27) which tends to zero as $r \to \infty$ can be expressed through the Whittaker function of the form

$$F(r) \sim W_{\beta,i\theta}(2\lambda r), \tag{29}$$

where

$$\beta = \frac{Ea}{\lambda}, \quad \theta = \sqrt{a^2 - (l+e_0B+1/2)^2} \tag{30}$$

near $E = -m$, or through the MacDonald function of imaginary order,

$$F(r) \sim \sqrt{r} K_{2i\theta}(\sqrt{8mar}), \tag{31}$$

for $E = -m$. It follows from Eqs. (29) or (31) that the bound electron state with $E \approx -m$ is localized in space. Such behavior of the electron state can be easily explained if we treat Eq. (27) as a Schrödinger equation with the effective energy $\epsilon = (E^2 - m^2)/2m$ and the effective potential

$$U_{\text{eff}}(r) = -\frac{2Ea}{mr} - \frac{a^2 - (l+e_0B)(l+e_0B+s)}{2mr^2}. \tag{32}$$

In the case $E \approx -m$ for the ground electron state $l = 0$ the potential $U_{\text{eff}}(r)$ has the form of a wide barrier. One notes that at large distances from the Coulomb center for an electron with energy $E \approx -m$ the effective potential is not attractive but repulsive. In the presence of the magnetic flux the height and width of the effective potential barrier increase for $s = 1$ and $e_0 B > 1$. Therefore, the probability of pair production by the Coulomb field decreases in the presence of the magnetic flux.

For a weak magnetic flux the form of the vector Coulomb potential for $r < R$ is not essential to the principal result. The calculation is most easily performed if we consider the simplest model with only the 3-vector Coulomb potential,

$$A^0(r) = \frac{a}{|e|R}, \quad A_r = 0, \quad A_\varphi = 0, \tag{33}$$

in the range $r \leq R$. Then the radial solution $F(r)$ that is finite at $r = 0$ in the range $r \leq R$ is expressed via the Bessel function of integer order $|l|$ as

$$F(r) \sim r J_{|l|}(cr), \tag{34}$$

where

$$c = \sqrt{\left(E + \frac{a}{R}\right)^2 - m^2}. \tag{35}$$

Applying the continuity relations

$$\left(\frac{G(r)}{F(r)}\right)_{r=R-0} = \left(\frac{G(r)}{F(r)}\right)_{r=R+0}, \tag{36}$$

and taking into account the fact that the parameter R must be small compared with $1/m$ and also that $E \approx -m$, we obtain the transcendental equation (for $l = 0$) that implicitly determines the energy spectrum of an electron near the negative-energy continuum boundary $-m$ as

$$cR\frac{J_1(cR)}{J_0(cR)} = 1 - \left(x\frac{W'_{\beta,i\theta}(x)}{W_{\beta,i\theta}(x)}\right)_{x=2\lambda R}, \tag{37}$$

and the critical charge for the ground state is determined by

$$2a_{cr}(B)\frac{J_1(a_{cr}(B))}{J_0(a_{cr}(B))} = 1 - \left(z\frac{K'_{i\nu}(z)}{K_{i\nu}(z)}\right)_{z=\sqrt{8ma_{cr}(B)R}}. \tag{38}$$

Here

$$\beta = -\frac{ma}{\lambda}, \quad \theta = \sqrt{a^2 - (e_0B + 1/2)^2}, \quad \nu = 2\sqrt{a_{cr}^2(B) - (e_0B + 1/2)^2}, \tag{39}$$

and the prime denotes differentiation with respect to the argument x of the Whittaker function in (37), or the argument z of the MacDonald function in (38).

Equations (37) and (38) can only be solved numerically. These equations can be simplified somewhat for $Rm \ll 1$. For instance, to simplify Eq. (38), we represent the MacDonald function in the form [19]

$$K_\mu(z) = \frac{\pi}{2\sin(\mu\pi)}[I_{-\mu} - I_\mu], \tag{40}$$

where

$$I_\mu = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\mu+k+1)} \left(\frac{z}{2}\right)^{\mu+2k}. \tag{41}$$

Keeping only the lowest order terms in the expansion of the MacDonald function at small values of the argument, we obtain

$$K_{i\nu}(z) \sim -\left(\frac{\pi}{\nu\sinh(\nu\pi)}\right)^{1/2} \sin\left(\nu\ln(|z|/2) + \arg\Gamma(1-i\nu)\right), \tag{42}$$

where $\Gamma(z)$ is the Euler gamma function. With Eq. (42), we finally obtain an approximate form of Eq. (38) at small Rm:

$$2a_{crb}\frac{J_1(a_{crb})}{J_0(a_{crb})} = 1 - \nu\cot\left[\nu\ln(|z|/2) + \arg\Gamma(1-i\nu)\right]. \tag{43}$$

Here $a_{crb} \equiv a_{cr}(B)$.

Numerical solution of Eq. (43) gives $a_{crb} \approx 0.79$ for $B = 0$, $a_{crb} \approx 0.84$ for $e_0B = 0.1$, $a_{crb} \approx 0.91$ for $e_0B = 0.2$ at $Rm = 0.02$, and $a_{crb} \approx 0.70$ for $B = 0$, $a_{crb} \approx 0.77$ for $e_0B = 0.1$, and $a_{crb} \approx 0.84$ for $e_0B = 0.2$ at $Rm = 0.006$. One sees that the critical charge increases with the increase of B and decreases with the decrease of Rm. Therefore, the instability of the vacuum of quantum electrodynamics in a strong vector Coulomb field in 2+1 dimensions in the presence of the magnetic flux tube of a very small radius must occur at a larger critical charge, as compared to that in the absence of the magnetic flux. Thus, the magnetic flux stabilizes the vacuum of quantum electrodynamics in the vector Coulomb field.

Similarly, Eq. (37) can be simplified. With the condition $Rm \ll 1$ we use the representation of the Whittaker function for small values of the argument in the form

$$W_{\beta,i\theta}(x) = \frac{\Gamma(2i\theta)}{\Gamma(1/2-\beta+i\theta)}x^{1/2-i\theta} + \frac{\Gamma(-2i\theta)}{\Gamma(1/2-\beta-i\theta)}x^{1/2+i\theta} \tag{44}$$

$$= \frac{|\Gamma(2i\theta)|}{|\Gamma(1/2-\beta+i\theta)|}2x^{1/2}\cos(\Phi(x)), \tag{45}$$

where

$$\Phi(x) = -\theta\ln x + \arg\Gamma(2i\theta) - \arg\Gamma(1/2-\beta+i\theta). \tag{46}$$

Taking the derivative with respect to x in Eq. (45) and substituting the resulting expression and function (45) in Eq. (37), we finally obtain the simplified transcendental equation

$$-\theta\ln(2\lambda R) + \arg\Gamma(2i\theta) - \arg\Gamma(1/2-\beta+i\theta) = \tan^{-1}Y + \pi n_r, \tag{47}$$

where

$$Y = \theta^{-1}\left(\frac{1}{2} - cR\frac{J_1(cR)}{J_0(cR)}\right). \tag{48}$$

The equation for the energy spectrum for any integer l can be derived in a similar way.

Equation (47) with $a < a_{crb}$ implicitly determines the eigen-spectrum of bound electron states for $l = 0$ with $m > E > -m$. It can be shown there are real solutions of this equation only for $a < a_{crb}$, but with $a > a_{crb}$ there is a formal solution of the form $E = E_0 - iw$, where $E_0 = -m - c_1(a - a_{crb})$, $c_1 \sim 1$ for the lowest state. With $a - a_{crb} \ll a_{crb}$, the imaginary part w is exponentially small. Such a solution can be found from formula (47) through analytic continuation of E as a function of a into the range $a > a_{crb}$. However, for $a - a_{crb} \ll a_{crb}$, the imaginary part can be more readily determined in another way. Indeed, the appearance of the imaginary part means that for $a > a_{crb}$, the corresponding Dirac equation has only a formal solution with $E = E_0 - iw$ for the electron states. However, for $a > a_{crb}$, the same equation also describes a positron with the energy $E_0 = m + c_1(a - a_{crb})$, since the Dirac equation for a positron for $a > a_{crb}$ can be obtained from the Dirac equation for an electron by replacing E with $-E$ and a with $-a$. Hence, for $a > a_{crb}$, the positron states are quasi-stationary. For $a - a_{crb} \equiv \Delta a \ll a_{crb}$, the width of the quasi-stationary level w can be estimated in the semiclassical approximation. To obtain this estimate, we must compute the transmission coefficient through the potential barrier in Eq. (27). We note that the width w is half of the reciprocal of the positron lifetime or twice the probability of pair creation by the Coulomb field. If $\Delta a \ll a_{crb}$, the positrons created by the field are very slow, and the Coulomb barrier is hardly transparent for them. The probability of pair creation is therefore exponentially small, i.e.,

$$w \sim m \exp\left[-2\pi a\left(\frac{m}{\sqrt{E^2 - m^2}} - 1\right)\right] \cong m \exp\left(-c_2\sqrt{\frac{a_{crb}}{\Delta a}}\right), \quad c_2 \sim 1. \tag{49}$$

IV. SUMMARY

In this paper we solve exactly the Dirac equation in 2+1 dimensions in a combination of Lorentz 3-vector, scalar Coulomb, and AB potentials. We employ the two-component Dirac equation which contains a new parameter introduced by Hagen to describe the spin of the spin-1/2 particle. We study the energy spectrum of an electron near the negative-energy continuum boundary and the critical charges for some electron states. Fermion pair production from a vacuum by a strong Coulomb field in the presence of the magnetic flux tube of zero radius is also considered. It is shown that the presence of the AB flux tends to stabilize the system.

Finally, we note that solutions to the two-component Dirac equation in the AB potential coincide with the solutions of the Dirac equation in 2+1 dimensions for a massive neutral fermion with the anomalous magnetic moment in a point charge placed at the origin $z = 0$ (see, for example, Ref. [20]). In the three-dimensional space, such a field corresponds to the electric field of a thin thread that is perpendicular to the plane $z = 0$ and carries the electric charge with a constant linear density. Thus, the solutions to the two-component Dirac equation in the AB potential can be directly applied to the planar scattering of a massive neutral fermion with anomalous magnetic moment interacting in the electric field of the thin thread, which was first predicted by Aharonov and Casher in Ref. [21].

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